

Free field construction of Heterotic string compactified on
Calabi-Yau manifolds of Berglund-Hubsch type in the
Batyrev-Borisov combinatorial approach

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Alexander Belavin,

Landau Institute for Theoretical Physics

Abstract and PREAMBLE

Heterotic string models in 4-dimensions are Hybrid theories of Left-moving $N = 1$ fermionic string whose additional 6-dimensions are compactified on a $N = 2$ SCFT theory with the central charge 9, and of Right-moving bosonic string, whose additional dimensions are also compactified on $N = 2$ SCFT theory with the central charge 9, and whose remaining 13 dimensions form the torus of $E(8) \times SO(10)$ Lie algebra.

The important class of exactly solvable Heterotic string models considered earlier by D. Gepner corresponds to the products of $N = 2$ minimal models with the total central charge $c = 9$. These models are known to describe Heterotic string models compactified on Calabi-Yau manifolds, which belong a special subclass of general CY manifolds of Berglund-Hubsch type.

We generalize this construction to all cases of compactifications on Calabi-Yau manifolds of general Berglund-Hubsch type, using Batyrev-Borisov combinatorial approach.

We show how the number of $27, \overline{27}$ representations of $E(6)$ and Singlet representations of $E(6)$ are determined by the data of reflexive Batyrev polytopes that determines considered CY-manifold.

Introduction

The Gepner construction of Heterotic string models starts as a product of two CFTs with the critical central charges.

Namely, the models are the product of Left-handed Fermionic string obtained from an $N = 1$ CFT whose extra 6 dimensions are compactified into a product of Minimal $N = 2$ SCFT models with a total central charge of 9,

and Right-handed Bosonic string whose extra 9 dimensions are also compactified onto a product of minimal models.

The remaining 13 Right dimensions form a torus of the algebra $E(8) \times SO(10)$.

As a result these models obtain the $N = 1$ Space-Time Supersymmetry, arising from GSO reduction on Left side,

and the Gauge symmetry $E(8) \times E(6)$ Lie algebra (whose subalgebra is $SU(3) \times SU(2) \times U(1)$) arising from a similar reduction on Right side.

Introduction

This work is a generalisation of Gepner construction. We construct the models of Heterotic string compactified on the product of a torus of $E(8) \times SO(10)$ Lie algebra and on Calabi-Yau manifolds of general Berglund-Hubsch types.

The method uses the Batyrev-Borisov combinatorial approach, to implement a vertex algebra realized by free bosonic and fermionic fields for the states of the Calabi-Yau sector.

The construction uses the requirement for the simultaneous fulfillment of Mutual locality of the Left-moving vertices with the $N = 1$ Spacetime SUSY generators and of Right-moving vertices with generators of $E(8) \times E(6)$ -Gauge symmetry.

After this, the requirement of mutual locality of the products of the Left and Right vertices of physical states with each other together with other requirements of the Conformal bootstrap, leads to a self-consistent result precisely for the above-chosen torus $E(8) \times SO(10)$.

Based on these requirements, we explicitly construct Vertex operators of the physical states of the theory in the following order.

The original Conformal field theory

The construction of Heterotic string starts with a theory that is the product of two Conformal Field theories:

$N = 1$ SCFT on the Left, holomorphic side and

$N = 0$ CFT on the Right, antiholomorphic side.

In Left sector we have the product of the 4-dimensional $N = 1$ CFT for the subsector of Space-Time with central charge 6, and the $N = 1$ CFT for the compact Calabi-Yau subsector with central charge 9, so that the total central charge in the left sector is $c = 15$.

$N = 1$ SCFT of Left ST subsector is the theory of 4 free bosons $x^\mu(z)$ and 4 Majorana fermions $\psi^\mu(z)$.

$N = 1$ SCFT of Left CY subsector with central charge 9 is the theory of free bosons $X_i^\pm(z)$ and Majorana fermions $\Psi_i^\pm(z)$, where $i = 1, \dots, 5$.

The algebra of vertex operators in this subsector is determined by the data of a Calabi-Yau manifold of Berglund-Hubsch type.

Left-moving energy-momentum tensor is (others will be defined later)

$$T^L(z) = \frac{1}{2} \eta_{\mu\nu} \partial x^\mu \partial x^\nu + \frac{1}{2} \eta_{\mu\nu} \psi^\mu(z) \partial \psi^\nu(z) + T_{CY}^L(z).$$

The original Conformal field theory

In Right sector we have the product of four $N = 0$ Conformal Field theories:

CFT Space-Time subsector with central charge 4, realized by 4 bosonic fields $\bar{x}^\mu(\bar{z})$,

CFT $E(8)$ subsector with the central charge 8, realized by 8 free boson fields $Y_I(\bar{z})$, $I = 1, \dots, 8$ compactified on the torus of the $E(8)$ algebra,

CFT $SO(10)$ subsector with the central charge 5, realized by 5 free bosonic fields $\Phi_\alpha(\bar{z})$, $\alpha = 1, \dots, 5$ compactified on the torus of the $SO(10)$ algebra,

CFT Calabi-Yau subsector with central charge 9.

So that the total central charge in the right sector is $c = 26$.

The CFT of CY subsector with the central charge 9 is realized similarly to the left one by free bosons $\bar{X}_i^\pm(\bar{z})$ and Majorana fermions $\bar{\Psi}_i^\pm(\bar{z})$, where $i = 1, \dots, 5$.

Right-moving energy-momentum tensor is

$$T^R(\bar{z}) = \frac{1}{2} \eta_{\mu\nu} \bar{\partial} \bar{x}^\mu \bar{\partial} \bar{x}^\nu + \frac{1}{2} (\bar{\partial} Y_I)^2 + \frac{1}{2} (\bar{\partial} \Phi_\alpha)^2 + T_{CY}^R(\bar{z}).$$

Calabi-Yau manifolds and N=2 SCFT

Following to Borisov we construct of N=2 SCFT models corresponding to Calabi-Yau models of the Berglund-Hubsch type in terms the free fields.

The N = 2 Virasoro superalgebra generators in this construction are expressed in terms of the free bosons $X_i^\pm(z)$, the Majorana fermions $\Psi_i^\pm(z)$, where $i = 1, \dots, 5$.

We will also use the free boson fields $H_i, i = 1, \dots, 5$, in terms of which the Majorana fermions are expressed as $\Psi_i^\pm(z) = \exp[\pm iH_i(z)]$.

The operator product expansion (OPE) of these fields looks as follows

$$\begin{aligned}X_i^+(u)X_j^-(z) &= \delta_{ij} \log(u-z) + \dots, \\ \psi_i^+(u)\psi_j^-(z) &= \delta_{ij}(u-z)^{-1} + \dots, \\ H_i(u)H_j(z) &= -\delta_{ij} \log(u-z) + \dots\end{aligned}$$

N= 2 Super-Virasoro algebra

The currents $T_{CY}(z)$, $G_{CY}^{\pm}(z)$ and the U(1) current $J_{CY}(z)$ that form N = 2 Virasoro algebra of Calabi-Yau subsectors look as follows

$$T_{CY}(z) = \sum_{i=1}^5 [\partial X_i^+ \partial X_i^- + \frac{1}{2}(\partial \Psi_i^+ \Psi_i^- + \partial \Psi_i^- \Psi_i^+) + \frac{1}{2}(a_i^+ \partial^2 X_i^- + a_i^- \partial^2 X_i^+)],$$

$$G_{CY}^+(z) = \sqrt{2} \sum_{i=1}^5 [(\Psi_i^+ \partial X_i^- + \partial \Psi_i^+ a_i^-)],$$

$$G_{CY}^-(z) = \sqrt{2} \sum_{i=1}^5 [(\Psi_i^- \partial X_i^+ + \partial \Psi_i^- a_i^+)],$$

$$J_{CY}(z) = \sum_{i=1}^5 [\Psi_i^+ \Psi_i^- + a_i^+ \partial X_i^- - a_i^- \partial X_i^+].$$

It is also be useful to represent the currents $T_{CY}(z)$ and $J_{CY}(z)$ in the following equivalent bosonic form

$$T_{CY}(z) = \sum_{i=1}^5 [\partial X_i^+ \partial X_i^- - \frac{1}{2}(\partial H_i)^2 + \frac{1}{2}(a_i^+ \partial^2 X_i^- + a_i^- \partial^2 X_i^+)],$$

$$J_{CY}(z) = \partial \sum_{i=1}^5 [iH_i + a_i^+ X_i^- - a_i^- X_i^+] = \partial H_{CY}(z).$$

Calabi-Yau manifolds and N=2 SCFT

The vectors \vec{a}^+ and \vec{a}^- depend on the Calabi-Yau manifold selected for compactification on it.

Below we will show how to do this, and also that the scalar product of these vectors (\vec{a}^+, \vec{a}^-) is equal 1.

The central charge c of the $N = 2$ Virasoro algebra under consideration is expressed through these vectors as follows

$$\frac{c}{3} = 5 - 2 \sum_{i=1}^5 a_i^+ a_i^- .$$

These statements will be explained below.

Thus, the central charge of the left and right Calabi-Yau subsectors is 9, which is necessary for Heterotic string theory to be self-consistent.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

Calabi-Yau manifolds of BH type are defined as a hypersurface (or its orbifold) in the weighted projective space $P_{\vec{k}}$ defined by the equation

$$W(y_1, \dots, y_5) = \sum_{i=1}^5 \prod_{j=1}^5 y_j^{A_{ij}} = 0.$$

Here $\vec{k} = k_1, \dots, k_5$, where $k_i, i = 1, \dots, 5$ are the weights of $P_{\vec{k}}$.

$W(y_1, \dots, y_5)$ is a nondegenerated polynomial with invertible integer matrix A_{ij} .

It is assumed that the variables y_i have positive rational degrees $q_i = \frac{k_i}{d}$, and $d = \sum_{i=1}^5 k_i$, such that $\sum A_{ij} q_j = 1$ for all i .

The polynomial W and the monomial $\prod_{i=1}^5 y_i$ are homogeneous with respect to the replacement $y_j \rightarrow \exp(i2\pi q_j) y_j$.

The symmetry group G described above always exists, since it simply follows from the self-consistency between the data of the projective space $P_{\vec{k}}$ and the polynomial W .

Let's call it the "minimum admissible group."

The maximum allowed group can be larger. How much exactly depends on the projective space of the data.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

The mirror CY-manifold is defined by the mirror polynomial W^T with the transposed matrix A_{ij}^T in the mirror projective space $P_{\bar{k}}^*$, and by the dual admissible group G^T .

This group will be determined automatically from the duality requirements.

From the described Berglund-Hubsch data, the potential W and the initial admissible group G , one can obtain the Batyrev-Borisov combinatorial data, which will be used in constructing the Calabi-Yau sectors in the Heterotic string.

This can be done as follows. Let M_0 and N_0 be two integer 5-dimensional lattices with bases \vec{u}_i and \vec{v}_j , whose pairing is

$$\vec{u}_i \cdot \vec{v}_j = A_{ij},$$

where A_{ij} are the exponents of the potential W .

The solution to these equations can be chosen as follows

$$(\vec{u}_i)_j = A_{ij}, \quad (\vec{v}_i)_j = \delta_{ij}.$$

We define vectors \vec{a}^+ and \vec{a}^- , which are related to vectors \vec{u}_i and \vec{v}_j as follows

$$\vec{a}^+ = \frac{1}{d^*} \sum_i k_i^* \vec{u}_i, \quad \vec{a}^- = \frac{1}{d} \sum_j k_j \vec{v}_j.$$

It is easy to verify that these vectors satisfy to the following equations

$$\vec{u}_i \cdot \vec{a}^- = 1, \quad \vec{a}^+ \cdot \vec{v}_j = 1.$$

Berglund-Hubsch type CY manifolds and N=2 SCFT models

Define the extension of the lattices M_0 and N_0 to a pair of dual Batyrev lattices M and N for the case when the group G for the polynomial W is minimal.

First, to perform this we extend the lattice N_0 to the lattice $N = N_0 + \vec{a}^-$. In the next step we find the basis of the lattice N .

We denote its elements as $\vec{e}_{\beta=1,\dots,5}^*$.

Then we find the basis of the dual lattice M as the five vectors \vec{e}_{α} whose pairing with \vec{e}_{β}^* , ($\alpha, \beta = 1, \dots, 5$) is

$$\vec{e}_{\alpha} \cdot \vec{e}_{\beta}^* = \delta_{\alpha,\beta}.$$

In what follows, an important role in the construction of heterotic strings will be played by the set of elements $\vec{m} \in M$ and $\vec{n} \in N$ belonging to M and N , as well as the reflexive Batyrev polyhedra Δ^+ and Δ^- inside these lattices.

The latter means means that $\vec{m} \in \Delta^+$ and $\vec{n} \in \Delta^-$, if

$$\vec{m} \cdot \vec{a}^- = 1, \quad \vec{a}^+ \cdot \vec{n} = 1.$$

It follows that the central charge in the subsector CY is equal to 9.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

In constructing the vertex algebra of Heterotic string we will use vertices corresponding to the points of the dual lattices M and N .

To construct fermion states we also need to use in both, Left and Right sectors vertices corresponding to $M \pm \frac{1}{2} \vec{a}^+$ and $N \pm \frac{1}{2} \vec{a}^-$ elements.

The reason for this extension is that we want to build not just a SCFT model, but Heterotic string theory that includes SCFT of CY as subsector.

Therefore, since we have a diagonal $N = 1$ SCFT in the left sector (and also in the space-time and CY subsectors), then in both its subsectors the vertex operators must simultaneously belong to either the Ramond (R) or the Neveu-Schwartz (NS) type.

For the theory to be consistent, we need both options. Moreover, in the absence of vertex algebras of both types (NS-type and R-type), we cannot obtain space-time SUSY.

A similar situation occurs in the right sector.

This will be explained below when we construct vertex algebras first in the left and then in the right sectors.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

Following to Borisov we define the Vertex algebra of Heterotic string as the cohomology in respect to the set differentials $D_{\vec{m}}$ and $D_{\vec{n}}$

$$D_{\vec{m}} = \oint du \vec{m} \cdot \vec{\Psi}^-(u) \exp(\vec{m} \cdot \vec{X}^-(u)) = \sum_{i=1}^5 m_i \oint du \exp(-iH_i + \vec{m} \cdot \vec{X}^-(u)),$$

$$D_{\vec{n}} = \oint du \vec{n} \cdot \vec{\Psi}^+(u) \exp(\vec{n} \cdot \vec{X}^+(u)) = \left(\sum_{i=1}^5 n_i \oint du \exp(iH_i + \vec{n} \cdot \vec{X}^+(u)), \right.$$

$$D_{\vec{m}}^2 = D_{\vec{n}}^2 = \{D_{\vec{m}}, D_{\vec{m}'}\} = \{D_{\vec{n}}, D_{\vec{n}'}\} = \{D_{\vec{m}}, D_{\vec{n}}\} = 0;$$

where $\vec{m} \in \Delta^+$ and $\vec{n} \in \Delta^-$.

The integrands in the definition of the differentials $D_{\vec{m}}$ and $D_{\vec{n}}$, as can be verified, are BRST-invariant total vertices of the Heterotic string.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

Calabi-Yau multiplier of any total vertex operator of Heterotic string contains an exponential factor of the following form

$$\exp \left(i \sum_{i=1}^5 S_i H_i + \vec{p} \cdot \vec{X}^- + \vec{q} \cdot \vec{X}^+ \right),$$

whose dimensions is

$$\Delta(\vec{S}, \vec{p}, \vec{q}) = \frac{1}{2} \vec{S}^2 + \vec{p} \cdot \vec{q} + \frac{1}{2} (\vec{p} \cdot \vec{a}^- + \vec{q} \cdot \vec{a}^+).$$

These CY factors of the vertex operators have the following properties.

Firstly, their vectors \vec{S} , \vec{p} , \vec{q} must be elements of three 5 dimensional lattices to satisfy the OPE axioms.

Secondly, their dimensions must be equal, as we see, to 0 or 1/2 modulo an integer in the NS-case, or equal to 3/8 modulo an integer in the R-case.

Thirdly, they must be mutually local with the differentials $D_{\vec{m}}$ and $D_{\vec{n}}$.

For this, the parameters \vec{S} , \vec{p} , \vec{q} must satisfy the requirement that the pairing (\vec{p}, \vec{a}^-) and (\vec{a}^+, \vec{q}) be integers or half-integers for all \vec{p} and \vec{q} .

It follows that if the vertex operator belongs to the NS-sector, then all S_i are integers, $\vec{p} \in M$ and $\vec{q} \in N$, and if the vertex operator belongs to the R-sector, then all S_i are half-integers, $\vec{p} \in M \pm \frac{1}{2} \vec{a}^+$ and $\vec{q} \in N \pm \frac{1}{2} \vec{a}^-$.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

For shortness, we will call the subset of states belonging to the intersection of the BRST cohomology set defined by the differential Q_{BRST} and the Borisov cohomology set defined by the differentials D_m and D_n the Space of quasi-physical states.

Below we find two additional reductions of this set that lead to an extension in Left sector of Poincaré symmetry to $N = 1$ Space-Time Supersymmetry, and in Right sector to an extension of $E(8) \times SO(10)$ to $E(8) \times E(8)$.

It is this set that will be the space of physical states of the Heterotic string theory.

Left-moving sector, $N=1$ SCFT

In Left sector we have the product of 4-dimensional $N = 1$ CFT for Left space-time subsector consisting of 4 bosons and 4 Majorano fermions with the central charge of 6 and

$N = 1$ SCFT for compact CY subsector with the central charge of 9, so that the total central charge in Left sector is $c = 15$.

The $N = 1$ CFT of Left Space-Time subsector is a theory of 4 free boson fields $x^\mu(z)$ and 4 Majorana fermion fields $\psi^\mu(z)$

$$x^\mu(z)x^\nu(0) = -\eta^{\mu\nu} \log z + \dots,$$

$$\psi^\mu(z)\psi^\nu(0) = \eta^{\mu\nu} z^{-1} + \dots,$$

As for the $N = 1$ CFT of Left Calabi-Yau subsector, this is the above-described theory of free bosons $X_i^\pm(z)$, fermions $\Psi_i^\pm(z)$ and free boson fields H_i , $i = 1, \dots, 5$ bosonizing the fermions.

Here the $N = 1$ symmetry corresponds to the subalgebra of the $N = 2$, which was defined above when representing the CY subsectors.

Left-moving sector, N=1 SCFT

The total left-moving $N = 1$ Virasoro algebra is the diagonal subalgebra in the direct sum of the Calabi-Yau compact subsector, introduced above and of the $N = 1$ Virasoro algebras of space-time degrees of freedom

$$T^L(z) = T_{\text{ST}}^L(z) + T_{\text{CY}}^L(z),$$

$$G^L(z) = G_{\text{ST}}^L(z) + G_{\text{CY}}^L(z),$$

$$T_{\text{ST}}^L = -\frac{1}{2} \partial x^\mu(z) \partial x_\mu(z) - \frac{1}{2} \psi^\mu(z) \partial \psi_\mu(z),$$

$$G_{\text{ST}}^L(z) = \partial x^\mu \psi_\mu(z),$$

$$G_{\text{CY}}^L(z) = G_{\text{CY}}^+ + G_{\text{CY}}^-(z).$$

This $N = 1$ Virasoro superalgebra action is correctly defined on the product of only NS-representations or on the product of only R-representations.

Left-moving sector, N=1 SCFT

We use BRST approach to define the physical states.

The BRST charge is given by the integral

$$Q_{\text{BRST}} = \oint dz [cT_{\text{mat}} + \gamma G_{\text{mat}} + \frac{1}{2}(cT_{\text{gh}} + \gamma G_{\text{gh}})],$$

where we introduced the ghost fields and N = 1 Virasoro superalgebra of the ghosts

$$\beta(z)\gamma(0) = -z^{-1} + \dots, \quad b(z)c(0) = z^{-1} + \dots$$

$$T_{\text{gh}} = -\partial bc - 2b\partial c - \frac{1}{2}\partial\beta\gamma - \frac{3}{2}\beta\partial\gamma,$$

$$G_{\text{gh}} = \partial\beta c + \frac{3}{2}\beta\partial c - 2b\gamma.$$

Left-moving sector, N=1 SCFT

The ghost space of states is characterized by the vacuum $V_q(z)$, which can be realized as a free scalar field exponent

$$V_q(z) = \exp(q\phi(z)), \quad \phi(z)\phi(0) = -\log(z) + \dots$$

The left-moving vertex can be written as

$$V_{\vec{\mu}}^L = P(\partial^k x^\mu, \partial^1 \tilde{H}_a, \partial^r H_i, \partial^s X_i^+, \partial^t X_i^-) \times \\ \times \exp\left(q\phi + i\lambda^a \tilde{H}_a + i \sum_{i=1}^5 S_i H_i + \vec{p}\vec{X}^- + \vec{q}\vec{X}^+ + i p_\mu x^\mu(z)\right).$$

Here P is a polynomial of the derivatives of the corresponding boson fields including the fields \tilde{H}_a , $a = 1, 2$.

These fields bosonize the Fermi fields of spacetime subsector as follows

$$\tilde{H}_a(z)\tilde{H}_b(0) = -\delta_{ab} \log(z) + \dots, \quad a, b = 1, 2. \\ \frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) = \exp[\pm i\tilde{H}_1], \quad \frac{1}{\sqrt{2}}(\psi^2 \pm i\psi^3) = \exp[\pm i\tilde{H}_2].$$

Left-moving sector, N=1 SCFT

The dimension of the left-moving vertex $V_{\vec{\mu}}^L$

$$\Delta^L(\vec{\mu}) = \Delta_{\text{gh}}(q) + \Delta_{\text{ST}} + \Delta_{\text{CY}}^L,$$

where

$$\Delta_{\text{gh}}(q) = -\frac{q(q+2)}{2},$$

$$\Delta_{\text{ST}}^L = \frac{1}{2}\vec{\lambda}^2,$$

$$\Delta_{\text{CY}}^L = \frac{1}{2}\vec{S}^2 + \vec{p} \cdot \vec{q} + \frac{1}{2}\vec{p} \cdot \vec{a}^- + \frac{1}{2}\vec{q} \cdot \vec{a}^+.$$

The phase of the vertices $V_{\vec{\mu}}^L(u)$ and $V_{\vec{\mu}'}^L(z)$

$$2\pi i \vec{\mu} \cdot \vec{\mu}' = \Delta^L(\vec{\mu} + \vec{\mu}') - \Delta^L(\vec{\mu}) - \Delta^L(\vec{\mu}').$$

It follows that

$$\vec{\mu} \cdot \vec{\mu}' = -qq' + \lambda \cdot \lambda' + \vec{S} \cdot \vec{S}' + \vec{p} \cdot \vec{q}' + \vec{p}' \cdot \vec{q}.$$

The vector λ^a in $\exp(i\lambda^a \tilde{H}_a)$ must satisfy the requirement of consistency with the structure $N = 1$ on the left side.

Therefore, in NS sector the vectors $\vec{\lambda}$ fall into the classes [0] and [V], and in R sector the vectors $\vec{\lambda}$ fall into the classes [S] and [C] of $SO(1,3)$ lattice.

The integrality of $\vec{\mu} \cdot \vec{\mu}'$ is the condition of mutual locality of two vertices.

Massless left movers and N=1 Space-Time supersymmetry

The vertices of massless physical states play two roles:

The first is that it is the states of this set that must correspond to the observed elementary particles.

The first is that it is the such states must correspond to the observed elementary particles.

The second is that some of such vertices can be used to extend the symmetry of the theory. Namely, they can be taken as currents whose integrals become additional generators for this extension.

Requiring the vertices to be cohomologies of Q_{BRST} , $D_{\vec{m}}$ and $D_{\vec{n}}$ simultaneously

in the NS-subsector we get the left-moving vertex of massless vector boson

$$\exp(-\phi(z))\psi^\nu(z)\exp(i p_\mu x^\mu(z)),$$

which can be rewritten using bosonization as

$$\exp\left(-\phi(z) + i\lambda^a \tilde{H}_a + i p_\mu x^\mu(z)\right), \vec{\lambda} = (\pm 1, 0), \vec{\lambda} = (0, \pm 1).$$

Also we find the left-moving vertex of massless scalar bosons

$$V_{\vec{m}} = \exp(-\phi + \vec{m} \cdot \vec{X}^- + i p_\mu x^\mu(z)),$$

$$V_{\vec{n}} = \exp(-\phi(z) + \vec{n} \cdot \vec{X}^+ + i p_\mu x^\mu(z)),$$

where $\vec{m} \in \Delta^+$ and $\vec{n} \in \Delta^-$.

Massless left movers and N=1 Space-Time supersymmetry

In Ramond subsector with canonical picture number $(-\frac{1}{2})$ we find the vertices of the massless spinors

$$J^\pm(\vec{\sigma}, \vec{S}) = \exp\left(-\frac{1}{2}\phi + i\vec{\sigma} \cdot \vec{H} + i\vec{S} \cdot \vec{H} \pm \frac{1}{2}(\vec{X}^+ \cdot \vec{a}^- - \vec{X}^- \cdot \vec{a}^+)\right) \exp(ip_\mu x^\mu(z)),$$

$$J^\pm(\dot{\vec{\sigma}}, \dot{\vec{S}}) = \exp\left(-\frac{1}{2}\phi + i\dot{\vec{\sigma}} \cdot \vec{H} + i\dot{\vec{S}} \cdot \vec{H} \pm \frac{1}{2}(\vec{X}^+ \cdot \vec{a}^- - \vec{X}^- \cdot \vec{a}^+)\right) \exp(ip_\mu x^\mu(z)),$$

$$J^\pm(\dot{\vec{\sigma}}, \vec{S}) = \exp\left(-\frac{1}{2}\phi + i\dot{\vec{\sigma}} \cdot \vec{H} + i\vec{S} \cdot \vec{H} \pm \frac{1}{2}(\vec{X}^+ \cdot \vec{a}^- - \vec{X}^- \cdot \vec{a}^+)\right) \exp(ip_\mu x^\mu(z)),$$

$$J^\pm(\vec{\sigma}, \dot{\vec{S}}) = \exp\left(-\frac{1}{2}\phi + i\vec{\sigma} \cdot \vec{H} + i\dot{\vec{S}} \cdot \vec{H} \pm \frac{1}{2}(\vec{X}^+ \cdot \vec{a}^- - \vec{X}^- \cdot \vec{a}^+)\right) \exp(ip_\mu x^\mu(z)),$$

where

$$\sigma^a = \pm \frac{1}{2}, \quad \sum_{a=1}^2 \sigma^a = \pm 1, \quad \dot{\sigma}^a = \pm \frac{1}{2}, \quad \sum_{a=1}^2 \dot{\sigma}^a = 0,$$

and

$$S_i = \pm \frac{1}{2}, \quad \sum_{i=1}^5 S_i = \frac{1}{2}, \quad \text{mod } 2 \quad \dot{S}_i = \pm \frac{1}{2}, \quad \sum_{i=1}^5 \dot{S}_i = -\frac{1}{2}, \quad \text{mod } 2 .$$

Massless left movers and N=1 Space-Time supersymmetry

All of these vertices are BRST cohomology, but the ones that are also cohomology of $D_{\bar{m}}$ and $D_{\bar{n}}$ are only the following vertices

$$J_{\sigma}^{+} = \exp \left(-\frac{1}{2} \phi + i \vec{\sigma} \cdot \vec{H} + \frac{1}{2} H_{CY}^L \right),$$

$$J_{\dot{\sigma}}^{-} = \exp \left(-\frac{1}{2} \phi + i \dot{\vec{\sigma}} \cdot \vec{H} - \frac{1}{2} H_{CY}^L \right),$$

$$J_{\dot{\sigma}}^{+} = \exp \left(-\frac{1}{2} \phi + i \dot{\vec{\sigma}} \cdot \vec{H} + \frac{1}{2} H_{CY}^L \right),$$

$$J_{\sigma}^{-} = \exp \left(-\frac{1}{2} \phi + i \vec{\sigma} \cdot \vec{H} - \frac{1}{2} H_{CY}^L \right),$$

where $H_{CY}^L(z) = \sum_{i=1} (iH_i + a_i^+ X_i^- - a_i^- X_i^+)$ and we have omitted the factors $\exp(i p_{\mu} x^{\mu}(z))$.

The first two currents J_{σ}^{+} and $J_{\dot{\sigma}}^{-}$ are mutually local, as are the other two currents.

We will use the first pair to extend Poincaré symmetry to $N = 1$ spacetime supersymmetry.

Massless left movers and N=1 Space-Time supersymmetry

We select the pair of currents $J^+(\vec{\sigma}, \vec{S})$ to determine N = 1 super-Poincaré supercharges as follows

$$Q_\sigma = \oint du J_\sigma^+(u) = \oint du \exp\left(-\frac{1}{2}\phi + i\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right),$$

$$Q_{\dot{\sigma}} = \oint du J_{\dot{\sigma}}^-(u) = \oint du \exp\left(-\frac{1}{2}\phi + i\dot{\vec{\sigma}} \cdot \vec{H} - \frac{1}{2}H_{CY}^L\right).$$

The supercharges Q_σ and $Q_{\dot{\sigma}}$ are spinors with respect to the Poincaré algebra and, together with the generators of this algebra, P_μ and $J_{\mu\nu}$, form N = 1 Poincaré superalgebra.

In order to obtain N = 1 Space-Time supersymmetry in the theory, we must leave from the vertices of $V_{\vec{\mu}_L}^{\vec{L}}$, where $\vec{\mu}_L = (q, \vec{\lambda}, Q_{CY}^L)$, which are the cohomologies of Q_{BRST} , D_m , D_n only those vertices that are mutually local with J_σ^+ and $J_{\dot{\sigma}}^-$.

These vertices are mutually local with the currents J_σ^+ and $J_{\dot{\sigma}}^-$ if

$$q + \sum_a \lambda^a + Q_{CY}^L \in 2\mathbb{Z}.$$

where $Q_{CY}^L = \sum_i S_i + \vec{p} \cdot \vec{a}^- - \vec{q} \cdot \vec{a}^+$ is U(1) charge of Calabi-Yau subsector. This equation is nothing more than GSO condition for the vertices in the Left sector. From GSO equation it follows that the total internal charges Q_{CY}^L of the vertices are integers or half-integers.

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This can be obtained explicitly. The vectors $\vec{\lambda}$ are weights of the algebra $SO(1, 3)$ (and $SO(2n)$), belonging to one of the four conjugacy classes of the weight lattice.

$$(0) : (0, 0, 0, \dots, 0) + \text{any root};$$

$$(V) : (1, 0, 0, \dots, 0) + \text{any root};$$

$$(S) : \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) + \text{any root};$$

$$(C) : \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) + \text{any root}.$$

From GSO equation it follows that in NS sector the vectors $\vec{\lambda}$ fall into classes [0] and [V] and in R sector the vector $\vec{\lambda}$ fall into classes[S] and [C] of $SO(1, 3)$. Therefore, we obtain the connection between the conjugacy class of $\vec{\lambda}$ and the $U(1)$ charge of the CY factor for all 4 cases of left vertices.

In the NS sector we get

$$Q_{CY}^L \in 2\mathbb{Z} + 1, \quad \vec{\lambda} \in [0],$$

$$Q_{CY}^L \in 2\mathbb{Z}, \quad \vec{\lambda} \in [V].$$

In the R sector we get

$$Q_{CY}^L \in 2\mathbb{Z} + \frac{1}{2}, \quad \vec{\lambda} \in [S],$$

$$Q_{CY}^L \in 2\mathbb{Z} - \frac{1}{2}, \quad \vec{\lambda} \in [C].$$

Right-moving sector, N=0 CFT

The space-time subsector of the Right-moving sector with the central charge 4 contains boson fields $\bar{X}^\mu(\bar{z})$.

To obtain in Right sector Bosonic string with the total central charge 26, we add boson fields $Y_I(\bar{z})$, $I = 1, \dots, 8$ which are compactified on the torus of the algebra $E(8)$ with the central charge 8,

we also add the bosons $\Phi_\alpha(\bar{z})$, compactified on the torus of the algebra $SO(10)$ with the central charge 5.

The final contribution is given by the right-moving part of the compact Calabi-Yau subsector with the central charge 9.

Then the energy-momentum tensor and general right-moving vertex are

$$\begin{aligned} \bar{T}_{\text{mat}}(\bar{z}) = & \frac{1}{2}(\eta_{\mu\nu}\bar{\partial}\bar{X}^\mu\bar{\partial}\bar{X}^\nu + (\bar{\partial}Y_I)^2 + (\bar{\partial}\Phi_\alpha)^2 + \times \\ & \times + \sum_{i=1}^5(\partial\bar{X}_i^+\partial\bar{X}_i^- + \frac{1}{2}(\partial\Psi_i^+\Psi_i^- + \partial\Psi_i^-\Psi_i^+) + \frac{1}{2}(a_i^+\partial^2\bar{X}_i^- + a_i^-\partial^2\bar{X}_i^+)). \end{aligned}$$

$$\bar{V}_{\vec{\mu}_R}^{\vec{l}}(\bar{z}) = P_{\text{gh}}(\bar{b}, \bar{c})P_{\text{st}}(\bar{\partial}\bar{X}^\mu)P_{\text{int}}(\bar{\partial}\bar{Y}^I, \bar{\partial}\Phi_\alpha, \bar{H}_i, \bar{X}_i^+, \bar{X}_i^-)$$

$$\exp(i\eta_I\bar{Y}^I + i\Lambda_\alpha\Phi^\alpha + i\sum_{i=1}^5 S_i\bar{H}_i + \vec{p}\cdot\vec{X}^- + \vec{q}\cdot\vec{X}^+ + i p_\mu X^\mu(\bar{z})),$$

$\vec{\eta}$ is vector of $E(8)$ root lattice, and \vec{l} is vector of the $SO(10)$ weight lattice.

Right-moving sector, N=0 CFT

Among the vertices we find, as in the left sector, BRST-invariant massless ones. First of all this is SO(1, 3) vector $V^\mu(\bar{z}) = \iota\bar{\partial}\bar{X}^\mu(\bar{z})$.

We also find the currents of E(8) algebra

$$J^I(\bar{z}) = \iota\bar{\partial}\bar{Y}^I(\bar{z}), \quad I = 1, \dots, 8, \quad J_{\vec{\epsilon}}(\bar{z}) = \exp[\iota\epsilon_I\bar{Y}^I](\bar{z}), \quad \vec{\epsilon}^2 = 2,$$

$$\vec{\epsilon} = \begin{cases} (\pm 1, \pm 1, 0, 0, 0, 0, 0, 0) + \text{permutations,} \\ (\pm \frac{1}{2}, \dots, \pm \frac{1}{2}) + \text{permutations, even number of } + \frac{1}{2}, \end{cases}$$

where the vectors $\vec{\epsilon}$ are the roots of E(8) algebra.

Also there are the currents of SO(10) algebra:

$$J_\alpha(\bar{z}) = \iota\bar{\partial}\Phi_\alpha(\bar{z}), \quad \alpha = 1, \dots, 5,$$

$$J_{\vec{\rho}}(\bar{z}) = \exp[\iota\rho_\alpha\Phi_\alpha](\bar{z}), \quad \rho_\alpha = \pm 1, \quad \sum(\rho_\alpha)^2 = 2,$$

where the vectors $\vec{\rho}$ are the roots of SO(10)

$$\vec{\rho} = (\pm 1, \pm 1, 0, 0, 0) + \text{permutations,}$$

and the current of U(1) algebra of the right Calabi-Yau sector

$$J_{\text{CY}}(\bar{z}) = \bar{\partial}\bar{H}_{\text{CY}}(\bar{z}), \quad \bar{H}_{\text{CY}}(\bar{z}) = \sum_{i=1} (i\bar{H}_i + a_i^+\bar{X}_i^- - a_i^-\bar{X}_i^+).$$

Right-moving sector, N=0 CFT

In the Right sector we find also massless SO(10) spinors

$$J_{\omega}^{\pm}(\bar{z}) = \exp(i\omega_{\alpha}\Phi_{\alpha} \pm \frac{1}{2}\bar{H}_{CY})(\bar{z})$$

$$\omega_{\alpha} = \pm\frac{1}{2}, \quad \sum \omega_{\alpha} = \frac{1}{2} \pmod{2\mathbb{Z}},$$

$$J_{\dot{\omega}}^{\pm}(\bar{z}) = \exp(i\dot{\omega}_{\alpha}\Phi_{\alpha} \pm \frac{1}{2}\bar{H}_{CY})(\bar{z}),$$

$$\dot{\omega}_{\alpha} = \pm\frac{1}{2}, \quad \sum \dot{\omega}_{\alpha} = -\frac{1}{2} \pmod{2\mathbb{Z}}.$$

The currents J_{ω}^{+} is mutually local with $J_{\dot{\omega}}^{-}$, and $J_{\dot{\omega}}^{-}$ is mutually local with J_{ω}^{+} . All these currents are cohomologies of the differentials $D_{\bar{m}}$ and $D_{\bar{n}}$.

By choosing one of these pairs (we choose $J_{\omega}^{+}(\bar{z})$, $J_{\dot{\omega}}^{-}$) and adding them to the currents of the algebra SO(10) and the algebra U(1), we extend the product of the algebra SO(10) \times U(1) to the algebra E(6).

Right-moving sector, N=0 CFT

One can rewrite the E_6 currents in terms of simple roots of E_6

$$\vec{\alpha}_1 = e_1 - e_2, \dots, \vec{\alpha}_4 = e_4 - e_5, \vec{\alpha}_5 = e_4 + e_5,$$

$$\vec{\alpha}_6 = -\frac{1}{2}(e_1 + \dots + e_5) + \frac{\sqrt{3}}{2}e_6,$$

where e_i are the orthonormal basic vectors in \mathbb{R}^6 .

So that the Cartan subalgebra currents are

$$J_j(\bar{z}) = \iota \vec{\alpha}_j \cdot \bar{\partial} \vec{H}(\bar{z}), \quad j = 1, \dots, 6, \text{ where}$$

$$\vec{H}(\bar{z}) = (\bar{H}_1(\bar{z}), \dots, \bar{H}_5(\bar{z}), \bar{H}_6(\bar{z})),$$

$$\bar{H}_6(\bar{z}) = \bar{H}_{CY}(\bar{z}) = \sum_{i=1} (i\bar{H}_i + a_i^+ \bar{X}_i^- - a_i^- \bar{X}_i^+).$$

The currents of ladder $E(6)$ operators are given by

$$E_j(\bar{z}) = \exp[\iota \vec{\alpha}_j \vec{H}(\bar{z})],$$

$$F_j(\bar{z}) = \exp[-\iota \vec{\alpha}_j \vec{H}(\bar{z})], \quad j = 1, \dots, 6.$$

$E(6)$ gauge symmetry (and not any other) in Right sector ensures self-consistency of the theory whose Left sector has $N = 1$ spacetime SUSY.

Remarkably, it is precisely these symmetries that are necessary for the Grand Unified Theory.

Right-moving sector, N=0 CFT

In the right sector we also find a massless boson vertex that belongs to the one-dimensional representation of SO(10)

$$V_{\vec{m}}^{[1]} = \exp(\bar{H}_{CY} + \vec{m} \cdot \vec{X}^-),$$

another massless boson vertex that belongs to the 10-dimensional representation of SO(10)

$$V_{\vec{m}}^{[10]} = \exp(i\vec{\Lambda} \cdot \vec{\Phi} + \vec{m} \cdot \vec{X}^-),$$

and one more massless boson vertex that belongs to the 16-dimensional spinor representation of SO(10)

$$V_{\vec{m}}^{[16]} = \exp(i(\vec{\Lambda} + \vec{\omega}) \cdot \vec{\Phi} + \frac{1}{2}\bar{H}_{CY} + \vec{m} \cdot \vec{X}^-),$$

where $\vec{m} \in \Delta^+$, $\vec{\Lambda} = (\pm 1, 0, 0, 0, 0) + \text{permutations}$ and $\vec{\Lambda} + \vec{\omega} = \vec{\tilde{\omega}}$.

These three types of vertices are the cohomologies of the differentials $D_{\vec{m}}$ and $D_{\vec{n}}$ and, as will be shown below, form a 27-dimensional representation of E(6). The right massless sector also contains $\tilde{27}$ representations of E(6), which are similarly constructed from dual representations of SO(10) whose vertices depend on $\vec{n} \in \Delta^-$. The numbers of representations 27 and $\tilde{27}$ are equal to the number of points in the reflexive Batyrev polytopes for a given CY-manifold.

Right-moving sector, N=0 CFT

Finally, the fourth class of massless right vertices, which are also BRST, $D_{\vec{m}}$ and $D_{\vec{n}}$ cohomology, are $E(8) \times E(6)$ singlets

$$V(\vec{S}, \vec{m}, \vec{n}) = \exp\left(i \sum_{i=1}^5 S_i H_i + \vec{m} \cdot \vec{X}^- + \vec{n} \cdot \vec{X}^+ + \nu p_\mu \bar{X}^\mu\right)(\bar{z}),$$

whose dimension

$$\Delta(\vec{S}, \vec{m}, \vec{n}) = \frac{\vec{S}^2}{2} + \vec{m} \cdot \vec{n} + \frac{\vec{m} \cdot \vec{a}^- + \vec{n} \cdot \vec{a}^+}{2} = 1,$$

and U(1) charges

$$Q_{CY}(\vec{S}, \vec{m}, \vec{n}) = \sum_i S_i + \vec{n} \cdot \vec{a}^- - \vec{m} \cdot \vec{a}^+ = 0.$$

Here $\vec{m} \cdot \vec{a}^- = \vec{n} \cdot \vec{a}^+ = 1$, that is $\vec{m} \in \Delta^+$ and $\vec{n} \in \Delta^-$.

Right-moving sector, N=0 CFT

There are three types of solutions to all the above requirements that are imposed on the $E(8) \times E(6)$ singlets.

The first type of the singlet vertices is

$$V(\vec{S}, \vec{m}, \vec{n}) = \exp(\vec{m} \cdot \vec{X}^- + \vec{n} \cdot \vec{X}^+ + \imath p_\mu \bar{X}^\mu)(\bar{z})$$

, where the vectors $\vec{m} \in \Delta^+$, $\vec{n} \in \Delta^-$ and their product $\vec{m} \cdot \vec{n}$ is equal to 0.

The second type of the singlet vertices is

$$V(\vec{S}, \vec{m}, \vec{n}) = \sum_i m_i \exp(-iH_i + \vec{m} \cdot \vec{X}^- + \imath p_\mu \bar{X}^\mu)(\bar{z})$$

, where $\vec{m} \in \Delta^+$.

The third type of the singlet vertices is

$$V(\vec{S}, \vec{m}, \vec{n}) = \sum_i n_i \exp(iH_i + \vec{n} \cdot \vec{X}^+ + \imath p_\mu \bar{X}^\mu)(\bar{z})$$

where $\vec{n} \in \Delta^-$.

The total number of singlet massless states determined by these conditions in the case of Quintic coincides with the result of D. Gepner

Massless right movers and E(6) gauge symmetry

To obtain the $E(8) \times E(6)$ gauge symmetry in the constructed model, we need to restrict the space of states in the right moving sector to a set of states compatible with the action of the $E(8) \times E(6)$ generators.

This means that we have to choose and keep among the vertices that are BRST cohomology, $D_{\vec{m}}$ and $D_{\vec{n}}$, only those that are mutually local with the currents $E(8)$ and $E(6)$.

These requirements are met if the following "GSO" consistency equation*s are satisfied

$$\vec{\rho} \cdot \vec{\Lambda} \in \mathbb{Z}, \quad \vec{\epsilon} \cdot \vec{\eta} \in \mathbb{Z},$$
$$\omega \cdot \Lambda + \frac{1}{2} Q_{CY}^R \in \mathbb{Z}.$$

Massless right movers and E(6) gauge symmetry

From the "GSO" equation*s in the right-moving sector we get that the SO(10) parts of the solutions of the right vertices, that fall into one of the four conjugacy classes of the SO(10) weight lattice, determine the sixth Calabi-Yau component Q_{CY}^{R} as follows

$$\vec{\Lambda} \in [0] \Rightarrow Q_{\text{CY}}^{\text{R}} \in 2\mathbb{Z},$$

$$\vec{\Lambda} \in [V] \Rightarrow Q_{\text{CY}}^{\text{R}} \in 2\mathbb{Z} + 1,$$

$$\vec{\Lambda} \in [S] \Rightarrow Q_{\text{CY}}^{\text{R}} \in 2\mathbb{Z} - \frac{1}{2},$$

$$\vec{\Lambda} \in [C] \Rightarrow Q_{\text{CY}}^{\text{R}} \in 2\mathbb{Z} + \frac{1}{2}.$$

The data of the above-considered vertices satisfy these conditions.

Mutual locality of the full physical vertices

The complete vertices of a Heterotic string must be left and right BRST invariant, obey the left GSO equation* and the right "GSO" equation*, be cohomologies of D_m and D_n , and be mutually local with respect to each other. We start our search of full mutually local vertices among the so called "quasi-diagonal". The "quasi-diagonal" full vertices are given by the following product of GSO-invariant left-moving and "GSO"-invariant right-moving factors

$$\exp(q\phi + \iota\vec{\lambda}\vec{H} + i\vec{S}_L \cdot \vec{H}_L + \vec{p}_L \vec{X}_L^- + \vec{q}_L \vec{X}_L^+)(z) \times \\ \times \exp(\iota\eta_I \vec{Y}^I + \iota\Lambda_\alpha \Phi^\alpha + i\vec{S}_R \vec{H}_R + \vec{p}_R \cdot \vec{X}_R^- + \vec{q}_R \cdot \vec{X}_R^+)(\bar{z}),$$

where

$$\vec{S}_L = \vec{S}_R, \quad \vec{p}_L = \vec{p}_R, \quad \vec{q}_L = \vec{q}_R.$$

The product of two such vertices after moving one around the other receives a complex factor, the phase of which has the following form

$$2\pi\iota(\vec{\mu}_L \cdot \vec{\mu}'_L - \vec{\mu}_R \cdot \vec{\mu}'_R) = 2\pi\iota(-q q' + \vec{\lambda} \cdot \vec{\lambda}' - \vec{\Lambda} \cdot \vec{\Lambda}').$$

Mutual locality of the full physical vertices

The mutually locality requirement

$$\vec{\mu}_L \cdot \vec{\mu}'_L - \vec{\mu}_R \cdot \vec{\mu}'_R \in \mathbb{Z}$$

imposes certain correlations between the classes $\vec{\lambda}$ and $\vec{\Lambda}$. The reason for this is the requirements of compatibility formulated above lead to a correlation between the picture number q , the conjugacy classes of $\vec{\lambda}$ and the CY $U(1)$ charge in the left sector. The same is true for the correlation between the $\vec{\Lambda}$ classes and the CY charge $U(1)$ in the right sector, which are required for compatibility with the $E(6)$ symmetry.

Taking into account also that for quasi-diagonal full vertices with $Q_{CY}^L - Q_{CY}^R \in \mathbb{Z}$, we get four types of them, which satisfy the following requirements

$$Q_{CY}^L, Q_{CY}^R \in 2\mathbb{Z} \Rightarrow \lambda \in [V], \Lambda \in [0],$$

$$Q_{CY}^L, Q_{CY}^R \in 2\mathbb{Z} + 1 \Rightarrow \lambda \in [0], \Lambda \in [V],$$

$$Q_{CY}^L, Q_{CY}^R \in 2\mathbb{Z} + \frac{1}{2} \Rightarrow \lambda \in [C], \Lambda \in [C],$$

$$Q_{CY}^L, Q_{CY}^R \in 2\mathbb{Z} - \frac{1}{2} \Rightarrow \lambda \in [S], \Lambda \in [S].$$

Mutual locality of the full physical vertices

These vertices are mutually local due to the proper correlation of the internal charges Q_{CY}^L , $SO(1,3)$ and the weights λ and $SO(10)$ weights Λ . However, this set of the full mutually "quasi-diagonal" vertices does not satisfy the requirement of space-time supersymmetry.

To solve the problem, we will use the fact that superpartners in the left sector obtained by action the $N = 1$ supercharges Q_σ and $Q_{\dot{\sigma}}$ satisfy, as can be checked, the all requirement on the left vertices defined above, including the requirement of space-time supersymmetry. So we solve the problem by simply adding these superpartners to the set of "quasi-diagonal" vertices.

A similar technique can be used on the right side of full vertices, where instead of $N = 1$ Poincaré supergenerators, the action of the $E(6)$ algebra generators is used to obtain $E(6)$ partners.

The result of these operations leads to the fulfillment of the requirement of $N = 1$ spacetime supersymmetry, $E(8) \times E(6)$ gauge symmetry and preservation of mutual locality of complete vertex operators.

Massless vertices in explicit form.

For phenomenological applications, the most important are the massless states of the Heterotic string, compactified to four dimensions.

In this section we explicitly represent complete vertices for massless physical states as products of suitable left and right vertices, where we omit the factor $\exp(i p_\mu x^\mu)$ in order to shorten notations.

The gravity supermultiplets

First, we obtain expressions for the vertices of the gravitational supermultiplet, including the graviton vertex

$$\exp(-\phi(z)) \psi^\mu(z) \times i\bar{\partial}x^\nu(\bar{z}).$$

The gauge supermultiplets

We also get the following expressions for the vertices of vector gauge supermultiplets. The currents of E(8) are

$$V_{\mu}^I(z, \bar{z}) = \exp(-\phi(z)) \psi^{\mu}(z) \times i\bar{\partial}\bar{Y}^I(\bar{z}), \quad I = 1, \dots, 8,$$
$$V_{\mu}^{\vec{e}}(z, \bar{z}) = \exp(-\phi(z)) \psi^{\mu}(z) \times \exp[i\epsilon_I \bar{Y}^I](\bar{z}),$$

where the vectors \vec{e} are the roots of E(8) algebra;

The vertices belonging to adjoint representation of SO(10) algebra

$$V_{\mu, \alpha}(z, \bar{z}) = \exp(-\phi(z)) \psi^{\mu}(z) \times i\bar{\partial}\Phi_{\alpha}(\bar{z}), \quad \alpha = 1, \dots, 5,$$
$$V_{\mu, \vec{\rho}}(z, \bar{z}) = \exp(-\phi(z)) \psi^{\mu}(z) \times \exp[i\rho_{\alpha} \Phi_{\alpha}](\bar{z}),$$

where the vectors $\vec{\rho}$ are the roots of SO(10);

The gauge supermultiplets

The vertex of U(1) algebra of the Calabi-Yau sector

$$V_{\mu}^{\text{CY}}(z, \bar{z}) = \exp(-\phi(z) \psi^{\mu}(z) \times \bar{\partial} \bar{H}_{\text{CY}}(\bar{z})),$$
$$\bar{H}_{\text{CY}}(\bar{z}) = \sum_{i=1} (i \bar{H}_i + a_i^+ \bar{X}_i^- - a_i^- \bar{X}_i^+).$$

We then extend the $E_8 \times \text{SO}(10) \times \text{U}(1)$ symmetry to $E_8 \times E_6$ using 32 spinor currents of SO(10) algebra J_{ω}^+ and $J_{\dot{\omega}}^-$ given by (29)

$$J_{\omega}^{\pm}(\bar{z}) = \exp(i\omega_{\alpha} \Phi_{\alpha} \pm \frac{1}{2} \bar{H}_{\text{CY}})(\bar{z})$$
$$\omega_{\alpha} = \pm \frac{1}{2}, \quad \sum \omega_{\alpha} = \frac{1}{2} \pmod{2\mathbb{Z}},$$
$$J_{\dot{\omega}}^{\pm}(\bar{z}) = \exp(i\dot{\omega}_{\alpha} \Phi_{\alpha} \pm \frac{1}{2} \bar{H}_{\text{CY}})(\bar{z}),$$
$$\dot{\omega}_{\alpha} = \pm \frac{1}{2}, \quad \sum \dot{\omega}_{\alpha} = -\frac{1}{2} \pmod{2\mathbb{Z}}.$$

As a result we obtain the 32 additional currents of E(6) algebra

$$V_{\omega}(z, \bar{z}) = \exp(-\phi(z) \psi^{\mu}(z) \times J_{\omega}^+(\bar{z})),$$
$$V_{\dot{\omega}}(z, \bar{z}) = \exp(-\phi(z) \psi^{\mu}(z) \times J_{\dot{\omega}}^-(\bar{z})),$$

that together with the other 46 currents of SO(10) and U(1) form the 78-dimensional adjoint representation of E(6) algebra.

27 and $\overline{27}$ supermultiplets

The third type are the vertex operators of 27 supermultiplet of E(6) algebra whose right-moving factors belong to the 1-dimensional, 10-dimensional and 16- dimensional representations of S(10) algebra. The left-moving factors of all these vertex operators belong to spinor representation of N = 1 Super Poincare algebra.

Thus, the vertex operators of the 27 supermultiplet include a 1-dimensional representation of SO(10)

$$V_{\vec{\sigma}, \vec{m}}(z, \bar{z}) = \exp\left(-\frac{1}{2}\phi + i\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right) \times \exp(H_{CY}^R + \vec{m} \cdot \vec{X}^-),$$

the 10-dimensional representation of SO(10)

$$V_{\vec{\sigma}, \vec{m}}^{\vec{\Lambda}}(z, \bar{z}) = \exp\left(-\frac{1}{2}\phi + i\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right) \times \exp(i\vec{\Lambda} \cdot \vec{\Phi} + \vec{m} \cdot \vec{X}^-),$$

and the 16-dimensional spinor representation of SO(10)

$$V_{\vec{\sigma}, \vec{m}}^{\dot{\omega}}(z, \bar{z}) = \exp\left(-\frac{1}{2}\phi + i\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right) (z) \times \exp\left(i\dot{\omega} \cdot \vec{\Phi} + \frac{1}{2}H_{CY}^R + \vec{m} \cdot \vec{X}^- \right),$$

where $\vec{m} \in \Delta^+$, $\vec{\Lambda} = (\pm 1, 0, 0, 0, 0) + \text{permutations}$.

27 and $\overline{27}$ supermultiplets

The condition $\vec{m} \in \Delta^+$ means that the number 27 of supermultiplets is equal to the number of dots \vec{m} of the Batyrev polyhedron corresponding to the Calabi-Yau manifold of the Heterotic string model under consideration. Explicit expressions for the other vertices of the 27 supermultiplet can be obtained by acting through OPE on these vertices by the generators of the $N = 1$ Poincaré superalgebra. The similar actions we obtain the set of vertices of the $\overline{27}$ supermultiplet. Note that this operation does not break the mutual locality between the extended set of vertex operators that was between the original ones.

Singlet supermultiplets

The final, fourth class of vertices that satisfies all the requirements of our construction is the $N = 1$ supermultiplets, which are $E(8) \times E(6)$ singlets. The first type of the singlet vertices is

$$V_{\vec{\sigma}, \vec{m}, \vec{n}}(z, \bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right) \times \exp(\vec{m} \cdot \vec{X}^- + \vec{n} \cdot \vec{X}^+),$$

where the vectors $\vec{m} \in \Delta^+$, $\vec{n} \in \Delta^-$ and their product $\vec{m} \cdot \vec{n} = 0$.

The second type of the singlet vertices is

$$V_{\vec{\sigma}, \vec{m}}(z, \bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right) \times \sum_i m_i \exp(-iH_i + \vec{m} \cdot \vec{X}^-)(\bar{z}),$$

where $\vec{m} \in \Delta^+$.

The third type of the singlet vertices is

$$V_{\vec{\sigma}, \vec{n}}(z, \bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma} \cdot \vec{H} + \frac{1}{2}H_{CY}^L\right) \times \sum_i n_i \exp(iH_i + \vec{n} \cdot \vec{X}^+),$$

where $\vec{n} \in \Delta^-$.

For the case when Calabi-Yau sector is defined by the Quintic polynomial we find a total number of $E(8) \times E(6)$ singlets, which is coincide with the result of Gepner.

Conclusion

In this paper we developed a method for explicitly constructing the models of Heterotic string compactified on the product of a torus of the Lie algebra $E(8) \times SO(10)$ and general Calabi-Yau manifolds of Berglund-Hubsch type. The construction uses Batyrev-Borisov combinatorial approach together with the use of Vertex algebra of free bosonic and fermionic operators to construct the Vertex algebra of the CY sector.

We used the requirement for the simultaneous fulfillment of mutual locality of the left-moving vertices with the space-time symmetry generators and of right-moving vertices with generators of $E(8) \times E(6)$ gauge symmetry together with the requirement of mutual locality of complete (left-right) vertices of physical states.

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